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ON THE PRODUCT OF AN ALTERNANT BY A SYMMETRIC FUNCTION.

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As is well known, the product of the simple alternant $|\, a_1^0\ a_2^1\ a_3^2\ a_4^3 \ldots \,|$ and a symmetric function of $a_1, a_2, a_3, \ldots$ is an aggregate of alternants. When the symmetric function is of the form $\Sigma\, a_1 a_2 a_3 \ldots a_r$, the product is a single alternant differing from the original in having each of the last exponents increased by unity. In the general case, the mode of obtaining the aggregate is a fairly simple problem. The problem of obtaining the coefficient of a given alternant without having at the same time to obtain the coefficients of all the alternants in the aggregate is a problem not so simple. As a partial solution of this problem Dr. Muir has shown how the coefficient of one term of the aggregate may be obtained independently of the others.

The object of this paper is to extend the general problem started by Muir.

It is apparent that if we have a table giving the coefficients of all alternants in the product, (1)

$$|\, 0123 \ldots (n-1) \,| \left(\sum a_1 a_2 a_3 \ldots a_{j_1}\right)^{i_1} \left(\sum a_1 a_2 a_3 \ldots a_{j_2}\right)^{i_2} \cdots \left(\sum a_1 a_2 a_3 \ldots a_{j_i}\right)^{i_h}$$

where $0 \leqq i_k \leqq n$, $1 \leqq j_h \leqq n$; $i_1 j_1 + i_2 j_2 + \ldots + i_h j_h = t \leqq n$; and where $i_1 + i_2 + i_3 + \ldots i_h = k$, we can by taking the proper multiples* of

$$(2)\quad \left(\sum a_1 a_2 a_3 \ldots a_{j_1}\right)^{i_1} \left(\sum a_1 a_2 a_3 \ldots a_{j_2}\right)^{i_2} \cdots \left(\sum a_1 a_2 a_3 \ldots a_{j_h}\right)^{i_h}$$

form any symmetric function S, and hence from such a table get the coefficients of any alternant in the product $|\, 0\ 1\ 2\ 3 \ldots (n-1) \,|\, S$.

*These multiples are found from the table of symmetric functions of weight t.

The difference equation for the coefficients of all alternants in the product (1) (where $i_1=1$, $j_1 \overline{\gtrless} \frac{1}{2}n$) is obtained and this is used to construct tables for certain cases.

1. Let $|\,0\;1\;2\;3 \ldots\ldots r_1 s_1 \ldots\ldots r_2 s_2 \ldots\ldots r_g s_g \ldots\ldots r_k s_k\,|$ denote the alternant wherein the numbers are consecutive except at the k points $r_1 s_1$, $r_2 s_2$, ..., $r_k s_k$,—that is, the numbers from 0 to r_1, s_1 to r_2, s_2 to r_3, and so on are consecutive; but $r_1 s_1$, $r_2 s_2$, etc., may differ by more than unity.

Let us denote the coefficient of this alternant in the product

$$|\,0123 \ldots\ldots (n-1)\,| \left(\sum a_1 a_2 a_3 \ldots\ldots a_{j_1}\right)\left(\sum a_1 a_2 a_2 \ldots\ldots a_{j_2}\right)^{i_2} \cdots\cdots \left(\sum a_1 a_2 a_3 \ldots\ldots a_{j_h}\right)^{i_h}$$

by

$$C^t \begin{Bmatrix} s_1 & s_2 & s_3 \ldots\ldots s_g \ldots\ldots s_k \\ 0 & 0 & 0 \ldots\ldots 0 \ldots\ldots 0 \end{Bmatrix} \tag{3}$$

and the coefficient of

$$|\,0\;1\;2\;3 \ldots\ldots r_1 (s_1-1) \ldots\ldots (s_1+\alpha-1)(s_1+\alpha) \ldots\ldots r_2 (s_2-1) s_2 \ldots\ldots (s_2+\beta-2)(s_2+\beta) \ldots\ldots$$
$$\ldots\ldots r_g (s_g-1) s_g \ldots\ldots (s_g+\gamma-2) \ldots\ldots r_k (s_k-1)\,|$$ in the product

$$|\,0123 \ldots (n-1)\,| \left(\sum a_1 a_2 a_3 \ldots a_{j_1}\right)\left(\sum a_1 a_2 a_3 \ldots a_{j_2}\right)^{i_2} \cdots \left(\sum a_1 a_2 a_3 \ldots a_{j_h}\right)^{i_h-1}$$

by

$$C^{t-j_h} \begin{Bmatrix} s_1 & s_2 \ldots\ldots s_g \ldots\ldots s_k \\ \alpha & \beta \ldots\ldots \gamma \ldots\ldots \kappa \end{Bmatrix},$$

where γ placed below s_g denotes that γ consecutive numbers beginning with s_g are decreased by unity.

2. Now it is easily seen that that the product

$$|\,0123 \ldots\ldots (n-1)\,| \left(\sum a_1 a_2 a_3 \ldots\ldots a_{j_1}\right)\left(\sum a_1 a_2 a_3 \ldots\ldots a_{j_2}\right)^{i_2} \cdots \left(\sum a_1 a_2 a_3 \ldots\ldots a_{j_h}\right)^{i_h-1}$$

$$= \ldots\ldots + \sum_0^{j_h} \alpha\beta \ldots\ldots \kappa\; C^{t-j_h} \begin{Bmatrix} s_1 & s_2 \ldots\ldots s_k \\ \alpha & \beta \ldots\ldots \kappa \end{Bmatrix} |\,0\;1\;2\;3 \ldots\ldots$$

$$\ldots\ldots r_1 (s_1-1) s_1 (s_1+1) \ldots\ldots (s_1+\alpha-2)(s_1+\alpha) \ldots\ldots r_g (s_g-1) s_g \ldots\ldots (s_g+\gamma-2)(s_g+\gamma) \ldots\ldots$$

$\ldots\ldots r_k (s_k-1)\,|$ + other terms, where $\alpha+\beta+\ldots\ldots+\kappa=j_h$; κ cannot of course be greater than 1.

Only those terms are written which on multiplying both sides by $\Sigma a_1 a_2 a_3 \ldots a_{j_h}$ can give rise to

$$|\,0\;1\;2\;3 \ldots\ldots r_1 s_1 \ldots\ldots r_2 s_2 \ldots\ldots r_g s_g \ldots\ldots r_k s_k\,|\;;$$

but the coefficient of this term in the product is (3). Hence this coefficient must be equal to the sum of the coefficients of the above terms, that is

$$C^t\begin{Bmatrix} s_1 & s_2 \ldots\ldots s_k \\ 0 & 0 \ldots\ldots 0 \end{Bmatrix} = \sum_{0}^{j_h} \alpha\,\beta\ldots\ldots\kappa \; C^{t-j_h}\begin{Bmatrix} s_1 & s_2 \ldots\ldots s_k \\ \alpha & \beta \ldots\ldots \kappa \end{Bmatrix}.$$

This is the relation between coefficients in the process of multiplication.

3. If we have a table giving the product of $|\,0\,1\,2\,3\ldots\ldots(n-1)\,|$ by all the symmetric functions of weight w, of the form (2), we can by means of the relation between coefficients found in article 2 construct a table giving the product of $|\,0\,1\,2\,3\ldots\ldots(n-1)\,|$ by all symmetric functions of the same type of weight $w+j_h$.

The following tables of this kind from weight one to weight seven have been constructed by means of the relation of article 2.

In this way we can construct the table of order t having those of order less than t.

	$\|0123\cdot(n-2)n\|$	$\|0123\cdot(n-3)(n-1)n\|$	$\|0123\cdot(n-2)(n+1)\|$	$\|0123\cdot(n-4)(n-2)(n-1)n\|$	$\|0123\cdot(n-3)(n-1)(n+1)\|$	$\|0123\cdot(n-2)(n+2)\|$	$\|0123\cdot(n-3)n(n+1)\|$	$\|0123\cdot(n-5)(n-3)(n-2)(n-1)n\|$	$\|0123\cdot(n-4)(n-2)(n-1)(n+1)\|$	$\|0123\cdot(n-3)(n-1)(n+2)\|$	$\|0123\cdot(n-2)(n+3)\|$	$\|0123\cdot(n-3)n(n+2)\|$	$\|0123\cdot(n-4)(n-2)n(n+1)\|$	$\|0123\cdot(n-6)(n-4)\cdots(n-1)n\|$	$\|0123\cdot(n-5)(n-3)(n-2)(n-1)(n+1)\|$	$\|0123\cdot(n-4)(n-2)(n-1)(n+2)\|$	$\|0123\cdot(n-3)(n-1)(n+3)\|$	$\|0123\cdot(n-2)(n+4)\|$
Σa_1	1																	
$(\Sigma a_1)^2$		1	1															
$\Sigma a_1 a_2$		1																
$(\Sigma a_1)^3$				1	2	1												
$\Sigma a_1 a_2 \Sigma a_1$				1	1													
$\Sigma a_1 a_2 a_3$				1														
$(\Sigma a_1)^4$							2	1	3	3	1							
$\Sigma a_1 a_2 (\Sigma a_1)^2$							1	1	2	1								
$(\Sigma a_1 a_2)^2$							1	1	1									
$\Sigma a_1 a_2 a_3 \Sigma a_1$								1	1									
$\Sigma a_1 a_2 a_3 a_4$								1										
$(\Sigma a_1)^5$												5	5	1	4	6	4	1
$\Sigma a_1 a_2 (\Sigma a_1)^3$												2	3	1	3	3	1	
$(\Sigma a_1 a_2)^2 \Sigma a_1$												1	2	1	2	1		
$\Sigma a_1 a_2 a_3 (\Sigma a_1)^2$													1	1	2	1		
$\Sigma a_1 a_2 a_3 \Sigma a_1 a_2$													1	1	1			
$\Sigma a_1 a_2 a_3 a_4 \Sigma a_1$														1	1			
$\Sigma a_1 a_2 a_3 a_4 a_5$														1				

	$\lvert 0123\cdots(n-4)(n-1)n(n+1)\rvert$	$\lvert 0123\cdots(n-5)(n-3)(n-2)n(n+1)\rvert$	$\lvert 0123\cdots(n-4)(n-2)n(n+2)\rvert$	$\lvert 0123\cdots(n-3)n(n+3)\rvert$	$\lvert 0123\cdots(n-3)(n+1)(n+2)\rvert$	$\lvert 0123\cdots(n-7)(n-6)\cdots(n-1)n\rvert$	$\lvert 0123\cdots(n-6)(n-4)\cdots(n-1)(n+1)\rvert$	$\lvert 0123\cdots(n-5)(n-3)(n-2)(n-1)(n+2)\rvert$	$\lvert 0123\cdots(n-4)(n-2)(n-1)(n+3)\rvert$	$\lvert 0123\cdots(n-3)(n-1)(n+4)\rvert$	$\lvert 0123\cdots(n-2)(n+5)\rvert$
$(\sum a_1)^6$ a	5	9	16	9	5	1	5	10	10	5	1
$\sum a_1a_2\,(\sum a)^4$	3	6	8	3	2	1	4	6	4	1	
$(\sum a_1a_2)^2(\sum a_1)^2$	2	4	4	1	1	1	3	3	1		
$(\sum a_1a_2)^3$	1	3	2		1	1	2	1			
$\sum a_1a_2a_3(\sum a_1)^3$	1	3	2			1	3	3	1		
$\sum a_1a_2a_3\sum a_1a_2\sum a_1$	1	2	1			1	2	1			
$(\sum a_1a_2a_3)^2$	1	1				1	1				
$\sum a_1a_2a_3a_4(\sum a_1)^2$		1				1	2	1			
$\sum a_1a_2a_3a_4\sum a_1a_2$		1				1	1				
$\sum a_1a_2a_3a_4a_5\sum a_1$						1	1				
$\sum a_1a_2a_3a_4a_5a_6$						1					

	$\lvert 0123\cdots(n-5)(n-3)(n-1)n(n+1)\rvert$	$\lvert 0123\cdots(n-6)(n-4)(n-3)(n-2)n(n+1)\rvert$	$\lvert 0123\cdots(n-4)(n-1)n(n+2)\rvert$	$\lvert 0123\cdots(n-5)(n-3)(n-2)n(n+2)\rvert$	$\lvert 0123\cdots(n-4)(n-2)n(n+3)\rvert$	$\lvert 0123\cdots(n-4)(n-2)(n+1)(n+2)\rvert$	$\lvert 0123\cdots(n-3)n(n+4)\rvert$	$\lvert 0123\cdots(n-3)(n+1)(n+3)\rvert$	$\lvert 0123\cdots(n-8)(n-6)\cdots(n-1)n\rvert$	$\lvert 0123\cdots(n-7)(n-5)\cdots(n-1)(n+1)\rvert$	$\lvert 0123\cdots(n-6)(n-4)\cdots(n-1)(n+2)\rvert$	$\lvert 0123\cdots(n-5)(n-3)\cdots(n-1)(n+3)\rvert$	$\lvert 0123\cdots(n-4)(n-2)(n-1)(n+4)\rvert$	$\lvert 0123\cdots(n-3)(n-1)(n+5)\rvert$	$\lvert 0123\cdots(n-2)(n+6)\rvert$
$(\sum a_1)^7$	14	14	21	35	35	21	14	14	1	6	15	20	15	6	1
$\sum a_1a_2(\sum a_1)^5$	9	10	11	20	15	10	4	5	1	5	10	10	5	1	
$(\sum a_1a_2)^2(\sum a_1)^3$	6	7	6	11	6	5	1	2	1	4	6	4	1		
$(\sum a_1a_2)^3\sum a_1$	4	5	3	6	2	3		1	1	3	3	1			
$\sum a_1a_2a_3(\sum a_1)^4$	4	6	3	8	3	2			1	4	6	4	1		
$\sum a_1a_2a_3\sum a_1a_2(\sum a_1)^2$	3	4	2	4	1	1			1	3	3	1			
$\sum a_1a_2a_3(\sum a_1a_2)^2$	2	3	1	2		1			1	2	1				
$(\sum a_1a_2a_3)^2\sum a_1$	2	2	1	1					1	2	1				
$\sum a_1a_2a_3a_4(\sum a_1)^3$	1	3		2					1	3	3	1			
$\sum a_1a_2a_3a_4\sum a_1a_2\sum a_1$	1	2		1					1	2	1				
$\sum a_1a_2a_3a_4\sum a_1a_2a_3$	1	1							1	1					
$\sum a_1a_2a_3a_4a_5(\sum a_1)^2$		1							1	2	1				
$\sum a_1a_2a_3a_4a_5\sum a_1a_2$		1							1	1					
$\sum a_1a_2a_3a_4a_5a_6\sum a_1$									1	1					
$\sum a_1a_2a_3a_4a_5a_6a_7$									1						

4. If we denote the product

$$| \, 0123 \ldots\ldots (n-1) \, | \left(\sum a_1 a_2 a_3 \ldots\ldots a_{j_2}\right)^{i_2} \left(\sum a_1 a_2 a_3 \ldots\ldots a_{j_3}\right)^{i_3} \ldots\ldots \left(\sum a_1 a_2 a_3 \ldots\ldots a_{j_h}\right)^{i_h - 1}$$

by P, then the alternant

$| \, 0123 \ldots\ldots r_1 s_1 \ldots\ldots r_2 s_2 \ldots\ldots r_g s_g \ldots\ldots r_k s_k \, |$ —which we shall represent temporarily by A_1— in the product $P \Sigma a_1 a_2 a_3 \ldots\ldots a_{j_1}$, (or P_1), can arise from the following alternants of P:

$$| \, 0\,1\,2\,3 \ldots\ldots r_1 (s_1 - 1) s_1 (s_1 + 1) \ldots\ldots (s_1 + \alpha_1 - 2)(s_1 + \alpha_1) \ldots\ldots r_g (s_g - 1)(s_g) \ldots\ldots$$

$$\ldots\ldots (s_g + \gamma_1 - 2)(s_g + \gamma_1) \ldots\ldots r_k (s_k - 1) \, | \, ,$$

where $\alpha_1, \beta_1, \ldots\ldots \gamma_1 \ldots\ldots \kappa_1 = 0, 1, 2, 3 \ldots\ldots j_1$,

and $\alpha_1 + \beta_1 + \gamma_1 \ldots\ldots \kappa_1 = j_1$.

The alternant

$$| \, 0\,1\,2\,3 \ldots\ldots (r_1 - j_h + 1) \ldots\ldots r_1 (r_1 + 1) s_1 \ldots\ldots r_2 s_2 \ldots\ldots r_g \, s_g \ldots\ldots r_k s_k \, | \, ,$$

which we shall also denote temporarily by A_2 in the product $P . \Sigma a_1 a_2 a_3 \ldots\ldots a_{j_1 + j_h}$, (or P_2), will evidently arise from the same terms of P, but it will arise also from the following terms:

$$| \, 0\,1\,2\,3 \ldots\ldots (r_1 - j_h + 1) \ldots\ldots r_1 (r_1 + 1) s_1 (s_1 + 1) \ldots\ldots (s_1 + a + \alpha_1 - 2)(s_1 + a + \alpha_1) \ldots\ldots$$

$$\ldots\ldots r_g \, s_g \, (s_g + 1) \ldots\ldots (s_g + a + \alpha_1 - 2)(s_g + a + \alpha_1) \ldots\ldots r_k (s_k - 1) \, | \, .$$

These terms could not exist, however, if $j_1 \overline{\gtrless} \tfrac{1}{2} n$.

5. If we denote the coefficient of A_2 by

$$C^{t}\overset{j_h}{\underset{r_1}{}} \begin{Bmatrix} s_1 & s_2 \ldots\ldots s_k \\ 0 & 0 \ldots\ldots 0 \end{Bmatrix}$$

where the j_h above the r_1 denotes that r_1 and the preceding $(j_h - 1)$ numbers are increased by unity.

Then, if $j_1 \overline{\gtrless} \tfrac{1}{2} n$, we will have

$$C^{t}\overset{j_h}{\underset{r_1}{}} \begin{Bmatrix} s_1 & s_2 \ldots\ldots s_k \\ 0 & 0 \ldots\ldots 0 \end{Bmatrix} = \sum_{0}^{j_h} {}_{\alpha \, \beta \ldots\ldots \kappa} \, C^{t - j_h}\overset{j_h}{\underset{r_1}{}} \begin{Bmatrix} s_1 & s_2 \ldots\ldots s_k \\ \alpha & \beta \ldots\ldots \kappa \end{Bmatrix} ;$$

for, under these conditions the coefficient of A_1 in the product P_1 is the same as the coefficient of A_2 in P_2.

This gives us a means of obtaining the coefficient of any alternant in the product $P_1 . \Sigma a_1 a_2 a_3 \ldots a_{j_h}$, provided we have all those in the product P_2; that is, we obtain coefficients in this table from others in the same table.

6. Having an incomplete table as is considered in the preceding article we could readily convert it into a complete table of order $(t-j_1)$ by removing in all possible ways j_1 numbers from the alternants in the product.

7. Let us now consider, in the product P_1:

$$| \, 0\,1\,2\,3 \ldots (n-1) \, | \sum a_1 a_2 a_3 \ldots a_{n-i} \left(\sum a_1\right)^i,$$

those alternants of the type

$$| \, 0\,1\,2\,3 \ldots r_1 s_1 \ldots r_2 s_2 \ldots r_3 s_3 \, |,$$

where $s_1 - r_1 = 2$, $s_2 - r_2 = 2$, and $s_3 - r_3 = q$.

This alternant in P_1 can evidently arise on multiplying

$$| \, 0\,1\,2\,3 \ldots r_1 s_1 \ldots\ldots r_2 s_2 \ldots\ldots r_3 (s_3 - 1) \, |.$$

$$| \, 0\,1\,2\,3 \ldots r_1 s_1 \ldots\ldots r_2 (s_2 - 1)(s_2 + 1) \ldots r_3 s_3 \, |$$

$$| \, 0\,1\,2\,3 \ldots r_1 (s_1 - 1)(s_1 + 1) \ldots r_2 s_2 \ldots\ldots r_3 s_3 \, |$$

by Σa_1, in the product

$$| \, 0\,1\,2\,3 \ldots (n-1) \, | \sum a_1 a_2 a_3 \ldots a_{n-i} \left(\sum a_1\right)^{i-1},$$

so that we have

$$C^i \begin{Bmatrix} s_1 & s_2 & s_3 \\ 0 & 0 & 0 \end{Bmatrix} = \sum_{0}^{1} {}_{\alpha\beta\gamma} \, C^{i-1} \begin{Bmatrix} s_1 & s_2 & s_3 \\ \alpha & \beta & \gamma \end{Bmatrix} \quad (\alpha + \beta + \gamma = 1).$$

8. Since $n - 1 \overline{\gtrless} \frac{1}{2} n$, the conditions of articles 4 and 5 are satisfied and we have the following reduction formula for the construction of tables:

$$C^{i0}_{s_1} \begin{Bmatrix} s_2 & s_3 \\ 0 & 0 \end{Bmatrix} = C^{i-1\,1}_{s_1} \begin{Bmatrix} s_2 & s_3 \\ 0 & 1 \end{Bmatrix} + C^{i-1\,1}_{s_1} \begin{Bmatrix} s_2 & s_3 \\ 1 & 0 \end{Bmatrix} + C^{i-1\,0}_{s_1} \begin{Bmatrix} s_2 & s_3 \\ 0 & 0 \end{Bmatrix}.$$

In the following table

$$\overline{\sum} a_1 = | \, 0\,1\,2\,3 \ldots (n-1) \, | \sum a_1 a_2 a_3 \ldots a_{n-1} \left(\sum a_1\right)^1,$$

$$\left(\overline{\sum} a_1\right)^2 = | \, 0\,1\,2\,3 \ldots (n-1) \, | \sum a_1 a_2 a_3 \ldots a_{n-2} \left(\sum a_1\right)^2,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\left(\overline{\sum} a_1\right)^{11} = | \, 0\,1\,2\,3 \ldots (n-1) \, | \sum a_1 a_2 a_3 \ldots a_{n-11} \left(\sum a_1\right)^{11}.$$

G.P.O.

	$/123\cdots n/$	$/023\cdot (n-1)(n+1)/$	$/013\cdot (n-2)(n(n+1))/$	$/013\cdot (n-1)(n+2)/$	$/0124\ (n-3)(n-1)\,n(n+1)/$	$/0124\cdot (n-2)\,n(n+2)/$	$/0124\cdot (n-1)(n+3)/$	$/01235\ (n-4)(n-2)(n-1)\,n(n+1)/$	$/01235\cdot (n-3)(n-1)\,n(n+2)/$	$/01235\ (n-2)\,n(n+3)/$	$/01235\cdot (n-1)(n+4)/$	$/01\cdots 46\ (n-5)(n-3)\cdots n(n+1)/$	$/01\cdots 46\cdots (n-4)(n-2)(n-1)\,n(n+2)/$	$/01\cdots 46\cdots (n-3)(n-1)\,n(n-3)/$	$/01\cdots 46\cdots (n-2)\,n(n+4)/$	$/01\cdots 46\cdots (n-1)(n+5)/$	$/01\cdots 57\cdot (n-6)(n-4)\cdots n(n+1)/$	$/01\cdots 57\cdot (n-5)(n-3)\cdots n(n+2)/$	$/01\cdots 57\cdot (n-4)(n-2)(n-1)\,n(n+3)/$	$/01\cdots 57\cdot (n-3)(n-1)\,n(n+4)/$	$/01\cdots 57\cdot (n-2)\,n(n+5)/$	$/01\cdots 57\cdot (n-1)(n+6)/$	$/01\cdots 68\cdot (n-7)(n-5)\cdots n(n+1)/$	$/01\cdots 68\cdot (n-6)(n-4)\cdots n(n+2)/$	$/01\cdots 68\cdot (n-5)(n-3)\cdots n(n+3)/$	$/01\cdots 68\cdot (n-4)(n-2)(n-1)\,n(n+4)/$	$/01\cdots 68\cdot (n-3)(n-1)\,n(n+5)/$	$/01\cdots 68\cdot (n-2)\,n(n+6)/$	$/01\cdots 68\cdot (n-1)(n+7)/$
$(\overline{\Sigma a_1})^1$	1	1																											
$(\overline{\Sigma a_1})^2$	1	2	1	1																									
$(\overline{\Sigma a_1})^3$	1	3	3	3	1	2	1																						
$(\overline{\Sigma a_1})^4$	1	4	6	6	4	8	4	1	3	3	1																		
$(\overline{\Sigma a_1})^5$	1	5	10	10	10	20	10	5	15	15	5	1	4	6	4	1													
$(\overline{\Sigma a_1})^6$	1	6	15	15	20	40	20	15	45	45	15	6	24	36	24	6	1	5	10	10	5	1							
$(\overline{\Sigma a_1})^7$	1	7	21	21	35	70	35	35	105	105	35	21	84	126	84	21	7	35	70	70	35	7	1	6	15	20	15	6	1
$(\overline{\Sigma a_1})^8$	1	8	28	28	56	112	56	70	210	210	70	56	224	336	224	56	28	140	280	280	140	28	8	48	120	160	120	48	8
$(\overline{\Sigma a_1})^9$	1	9	36	36	84	168	84	126	478	478	126	126	504	756	504	126	84	420	840	840	420	84	36	216	540	720	540	216	36
$(\overline{\Sigma a_1})^{10}$	1	10	45	45	120	240	120	210	630	630	210	252	1008	1512	1008	252	210	1050	2100	2100	1050	210	120	720	1800	2400	1800	720	120
$(\overline{\Sigma a_1})^{11}$	1	11	55	55	165	330	165	330	990	990	330	462	1848	2772	1848	462	462	2310	4620	4620	2310	462	330	1980	4950	6600	4950	1980	330

9. We wish to prove the formula

$$C^{i\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0 \end{matrix}\right\} = \frac{i(i-1)\ (i-2)\ldots\ldots(i-s_1+2)}{\underline{|s_1-1}} \cdot \frac{(s_1-2)(s_2-3)\ldots\ldots(s_1-s_3+r_3-2)}{\underline{|s_3-r_3-1}}.$$

In the reduction formula of article 8, let i receive the suceessive values 2, 3, 4, 5.....i, and we have the following result:

$$C^{2\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0 \end{matrix}\right\} = C^{1\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 1 \end{matrix}\right\} + C^{1\,1}_{s_1}\left\{\begin{matrix} s_1 & s_2 \\ 1 & 0 \end{matrix}\right\} + C^{1\,0}_{s_1}\left\{\begin{matrix} s_2 & s_2 \\ 0 & 0 \end{matrix}\right\}$$

$$C^{3\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0 \end{matrix}\right\} = C^{2\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 1 \end{matrix}\right\} + C^{2\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 1 & 0 \end{matrix}\right\} + C^{2\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0 \end{matrix}\right\}$$

$$C^{4\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0 \end{matrix}\right\} = C^{3\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0 \end{matrix}\right\} + C^{3\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 1 & 0 \end{matrix}\right\} + C^{3\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0 \end{matrix}\right\}$$

$$C^{5\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0 \end{matrix}\right\} = C^{4\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 1 \end{matrix}\right\} + C^{4\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 1 & 0 \end{matrix}\right\} + C^{4\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0 \end{matrix}\right\}$$

....

$$C^{i-1\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0 \end{matrix}\right\} = C^{i-2\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 1 \end{matrix}\right\} + C^{i-2\,1}_{s_1}\left\{\begin{matrix} s_2 & s_2 \\ 1 & 0 \end{matrix}\right\} + C^{i-2\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0 \end{matrix}\right\}$$

$$C^{i\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0 \end{matrix}\right\} = C^{i-1\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 1 \end{matrix}\right\} + C^{i-1\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 1 & 0 \end{matrix}\right\} + C^{i-1\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0 \end{matrix}\right\}$$

$$C^{i\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0 \end{matrix}\right\} = \sum_{1}^{i-1}{}^{j}\, C^{j\,1}_{s_1}\left\{\begin{matrix} s_2 & s_2 \\ 0 & 1 \end{matrix}\right\} + \sum_{1}^{i-1}{}^{j}\, C^{j\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 1 & 0 \end{matrix}\right\}$$

10. Let the reduction formula, designated R,

$$C^{i\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0 \end{matrix}\right\} = \sum_{1}^{i-1}{}^{j}\, C^{j\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 1 \end{matrix}\right\} + \sum_{1}^{i-1}{}^{j}\, C^{j\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 1 & 0 \end{matrix}\right\}$$

operate upon itself and the successive results:

$$C^{i\,0}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 0 \end{Bmatrix} = C^{i-1\,1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 1 \end{Bmatrix} + C^{i-1\,1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 0 \end{Bmatrix}$$

$$C^{i-2\,1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 1 \end{Bmatrix} + C^{i-2\,1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 0 \end{Bmatrix}$$

$$C^{i-3\,1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 1 \end{Bmatrix} + C^{i-3\,1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 0 \end{Bmatrix}$$

$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$

$$C^{s_1-2\,1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 1 \end{Bmatrix} + C^{s_1-2\,1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 0 \end{Bmatrix}$$

Observe that the terms of R beyond the limit s_1-2 are zero.

$$=1\left(C^{i-2\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 2 & 0 \end{Bmatrix} + 2\,C^{i-2\,2}_{s_1}\begin{Bmatrix} s_2 & s_2 \\ 1 & 1 \end{Bmatrix} + C^{i-2\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 2 \end{Bmatrix}\right)$$

$$2\left(C^{i-3\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 2 & 0 \end{Bmatrix} + 2\,C^{i-3\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 1 \end{Bmatrix} + C^{i-3\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 2 \end{Bmatrix}\right)$$

$$(a) \qquad 3\left(C^{i-4\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 2 & 0 \end{Bmatrix} + 2\,C^{i-4\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 1 \end{Bmatrix} + C^{i-4\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 2 \end{Bmatrix}\right)$$

$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$

$$(i-2)\; C^{s_1-3\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 2 & 0 \end{Bmatrix} + 2\,C^{s_1-3\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 1 \end{Bmatrix} + C^{s_1-3\,2}_{s_1}\begin{Bmatrix} s & s_3 \\ 0 & 2 \end{Bmatrix}$$

$$=1\left(C^{i-3\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 3 & 0 \end{Bmatrix} + 3\,C^{i-3\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 2 & 1 \end{Bmatrix} + 3\,C^{i-3\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 2 \end{Bmatrix} + C^{i-3\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 3 \end{Bmatrix}\right)$$

$$3\left(C^{i-4\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 3 & 0 \end{Bmatrix} + 3\,C^{i-4\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 2 & 1 \end{Bmatrix} + 3\,C^{i-4\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 2 \end{Bmatrix} + C^{i-4\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 3 \end{Bmatrix}\right)$$

$$(b)\,6\left(C^{i-5\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 3 & 0 \end{Bmatrix} + 3\,C^{i-5\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 2 & 1 \end{Bmatrix} + 3\,C^{i-5\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 2 \end{Bmatrix} + C^{i-5\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 3 \end{Bmatrix}\right)$$

$$10\left(C^{i-6\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 3 & 0 \end{Bmatrix} + 3\,C^{i-6\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 2 & 1 \end{Bmatrix} + 3\,C^{i-6\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 2 \end{Bmatrix} + C^{i-6\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 3 \end{Bmatrix}\right)$$

$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$

$$\frac{(i-2)(i-3)}{|\underline{2}}\left(C^{s_1-4\,3}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 3 & 0\end{matrix}\right\}+3\,C^{s_1-4\,3}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 2 & 1\end{matrix}\right\}\right.$$

$$\left.+3\,C^{s_1-4\,3}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 1 & 2\end{matrix}\right\}+C^{s_1-4\,3}_{s_1}\left\{\begin{matrix} s_2 & c_3 \\ 0 & 3\end{matrix}\right\}\right)$$

....

....

$$=1\left(C^{i-s_1+2\,s_1-1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ s_1-2 & 0\end{matrix}\right\}+(s_1-2)\,C^{i-s_1+2\,s_1-1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ s_1-3 & 1\end{matrix}\right\}\right.$$

$$\left.+\frac{(s_1-2)(s_1-3)}{|\underline{2}}\,C^{i-s_1+2\,s_1-1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ s_1-4 & 2\end{matrix}\right\}+\ldots\right)$$

$$(s_1-2)\left(C^{i-s_1+1\,s_1-1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ s_1-2 & 0\end{matrix}\right\}+(s_1-2)\,C^{i-s_1+1\,s_1-1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ s_1-3 & 1\end{matrix}\right\}\right.$$

$$\left.+\frac{(s_1-2)(s_1-3)}{|\underline{2}}\,C^{i-s_1+1\,s_1-1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ s_1-4 & 2\end{matrix}\right\}+\ldots\right)$$

$$(c)\quad\frac{(s_1-2)(s_1-3)}{|\underline{2}}\left(C^{i-s_1\,s_1-1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ s_1-2 & 0\end{matrix}\right\}+(s_1-2)\,C^{i-s_1\,s_1-1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ s_1-3 & 1\end{matrix}\right\}\right.$$

$$\left.+\frac{(s_1-2)(s_1-3)}{|\underline{2}}\,C^{i-s_1\,s_1-1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ s_1-4 & 2\end{matrix}\right\}+\ldots\right)$$

$$\frac{(s_1-2)\,(s_1-3)\,(s_1-4)}{|\underline{3}}\left(C^{i-s_2-1\,s_1-1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ s_1-2 & 0\end{matrix}\right\}\right.$$

$$+(s_1-2)\,C^{i-s_1-1\,s_1-1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ s_1-3 & 1\end{matrix}\right\}$$

$$\left.+\frac{(s_1-2)(s_1-3)}{|\underline{2}}\,C^{i-s_2-1\,s_1-1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ s_1-4 & 2\end{matrix}\right\}+\ldots\right)$$

....

$$\frac{(i-2)\,(i-3)\ldots(i-s_1+2)}{|\underline{s_1-3}}\left(C^{1\,s_1-1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ s_1-2 & 0\end{matrix}\right\}+(s_1-2)\,C^{1\,s_1-1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ s_1-3 & 1\end{matrix}\right\}\right.$$

$$+\frac{(s_1-2)(s_1-3)}{\underline{|2}}C^{1}{}^{s_1-1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ s_1-4 & 2\end{Bmatrix}+\dots\dots+\dots\dots$$

$$\dots\dots+\frac{(s_1-2)(s_1-3)\dots\dots(s_1-k-1)}{\underline{|k}}C^{1}{}^{s_1-1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ s_1-k-2 & k\end{Bmatrix}+\dots\dots\Big).$$

In (c) let $i=i$, $s_1=s_1$, and $s_3=r_3+1$. Then

$$C^{i}{}^{0}_{s_1}\begin{Bmatrix} s_2 & (r_3+1) \\ 0 & 0\end{Bmatrix}=1C^{i-s_1+2}{}^{s_1-1}_{s_1}\begin{Bmatrix} s_2 & (r_3+1) \\ s_1-2 & 0\end{Bmatrix}+$$

$$(s_1-2)C^{i-s_1+1}{}^{s_1-1}_{s_1}\begin{Bmatrix} s_2 & (r_3+1) \\ s_1-2 & 0\end{Bmatrix}+\frac{(s_1-1)(s_1-2)}{\underline{|2}}C^{i-s_1}{}^{s_1-1}_{s_1}\begin{Bmatrix} s_2 & (r_3+1) \\ s_2-1 & 0\end{Bmatrix}$$

$$+\dots\dots+\frac{(i-2)\,(i-3)(i-4)\dots\dots(i-s_1+2)}{\underline{|s_1-3}}C^{1}{}^{s_1-1}_{s_1}\begin{Bmatrix} s_2 & (r_3+1) \\ s_1-1 & 0\end{Bmatrix}.$$

It is evident from the table that the C's have values, respectively, from $i-s_1+2$ down to 1. Wherefore

$$C^{i}{}^{0}_{s_1}\begin{Bmatrix} s_2 & (r_3+1) \\ 0 & 0\end{Bmatrix}=1.(i-s_1+2)+(s_1-2)(i-s_1+1)+\frac{(s_1-1)(s_1-2)}{\underline{|2}}(i-s_1)$$

$$+\frac{s_1(s_1-1)(s_1-2)}{\underline{|3}}(i-s_1-1)+\dots\dots+\frac{(i-2)(i-3)(i-4)\dots\dots(i-s_1+2)}{\underline{|s_1-3}}(i-(i-1))$$

$$=i\Big[1+(s_1-2)+\frac{(s_1-1)(s_1-2)}{\underline{|2}}+\frac{s_1(s_1-1)(s_1-2)}{\underline{|3}}+$$

$$\dots\dots+\frac{(i-2)(i-3)\dots\dots(i-s_1+2)}{\underline{|s_1-3}}\Big]-(s_1-2)\Big[1+(s_1-1)$$

$$+\frac{(s_1-1)s_1}{\underline{|2}}+\frac{(s_1-1)s_1(s_1+1)}{\underline{|3}}+\dots\dots+\frac{(i-1)(i-2)(i-3)\dots\dots(i-s_1+2)}{\underline{|s_1+2}}\Big].$$

Summing the two series in the brackets, we have

$$C^{i}{}^{0}_{s_1}\begin{Bmatrix} s_2 & (r_3+1) \\ 0 & 0\end{Bmatrix}=\frac{i(i-1)\,(i-2)\dots\dots(i-s_1+2)}{\underline{|s_1-2}}-(s_1-2)\frac{i(i-1)\dots\dots(i-s_1+2)}{\underline{|s_1-1}}$$

$$=\frac{i(i-1)(i-2)(i-3)\ldots\ldots(i-s_1+2)}{\lfloor s_1-1}(s_1-1-(s_1-2))$$

$$=\frac{i(i-1)(i-2)(i-3)\ldots\ldots(i-s_1+2)}{\lfloor s_1-1}.1.$$

In a similar manner, it can be proven that

$$C^{i}{}^{0}_{s_1}\left\{\begin{matrix} s_2 & r_3+2 \\ 0 & 0 \end{matrix}\right\}=\frac{i(i-1)(i-2)(i-3)\ldots\ldots(i-s_1+2)}{\lfloor s_1-1}(s_1-2)$$

$$C^{i}{}^{0}_{s_1}\left\{\begin{matrix} s_2 & r_3+3 \\ 0 & 0 \end{matrix}\right\}=\frac{i(i-1)(i-2)(i-3)\ldots\ldots(i-s_1+2)}{\lfloor s_1-1}\frac{(s_1-2)\ (s_1-3)}{\lfloor 2},$$

or, in general,

$$C^{i}{}^{0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0 \end{matrix}\right\}=\frac{i(i-1)(i-2)(i-3)\ldots\ldots(i-s_1+2)}{\lfloor s_1-1}$$

$$\times\frac{(s_1-2)\ (s_1-3)\ldots\ldots(s_1-s_3+r_3-2)}{\lfloor s_3-r_3-1}.$$

SYRACUSE UNIVERSITY, *December 18, 1902.*

AN ACCOUNT OF PROFESSOR RUNKLE'S MATHEMATICAL MONTHLY.

By PROFESSOR SIMON NEWCOMB.

I first made Mr. Runkle's acquaintance in the winter of 1857, when he was senior assistant in the Nautical Almanac Office, then at Cambridge, Mass. His intelligence, intellectual activity, and lively interest in matters and things generally, not excluding things political, made him a very interesting character.

It was early in 1858 that he announced to me and some others in the office his intention of starting a mathematical journal. His first step was to secure the necessary support. It may be feared that few in our day have an adequate conception of the backward condition of mathematical study in our country at that time. A curious illustration is offered by Davies' well-known dictionary of mathematics, in the preface of which it was announced that it contained defin-